MATH 5061 Lecture 9 (Mar 17)

[Note: PS 4 due date postponed to next Wed.]

Last time: Submanifold $M^k \subseteq (\bar{M}^n, \bar{g})$, $\bar{g}|_M =: g$

$\rightsquigarrow$ Study (M^k, g) from two perspectives: intrinsic OR extrinsic

Roughly speaking: $T_p\bar{M} = T_pM \oplus (T_pM)^\perp$

"differentiate g" $\rightsquigarrow$ $\bar{\nabla} = \bar{\nabla}^T + \bar{\nabla}^N$

Riem. conn. on $(\bar{M}, \bar{g})$; Riem. conn. on (M, g) ; 2^{nd} f.f. $A(X,Y) := (\bar{\nabla}_X Y)^N$

Equivalently, $\eta \in \Gamma(NM)$: $S_\eta(X) := -(\bar{\nabla}_X \eta)^T$

"differentiate g twice" $\rightsquigarrow$ $\exists$ 3 "constraint eq$^{\underline{s}}$"

Note: $\langle A(X,Y), \eta \rangle = \langle S_\eta(X), Y \rangle$

Gauss: $\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) - \langle A(Y,W), A(X,Z) \rangle + \langle A(X,W), A(Y,Z) \rangle$

Codazzi: $\bar{R}(X,Y,Z,\eta) = (\nabla_Y A)(X,Z,\eta) - (\nabla_X A)(Y,Z,\eta)$

Ricci: $\bar{R}(X,Y,\eta,\xi) = \langle R^\perp(X,Y)\eta, \xi \rangle + \langle [S_\eta, S_\xi](X), Y \rangle$

Consider the special case of $M^2 \subset (\mathbb{R}^3, g_{Eucl.})$

Fix $p \in M$, $\sigma = T_pM \subseteq T_p\mathbb{R}^3$ $\quad \sigma = \text{Span}\{e_1, e_2\}$ O.N.B.

$$\bar{R}(e_1,e_2,e_1,e_2) = R(e_1,e_2,e_1,e_2) - \langle A(e_2,e_2), A(e_1,e_1) \rangle + \langle A(e_1,e_2), A(e_2,e_1) \rangle$$

($\because \mathbb{R}^3$ is flat)

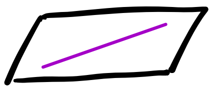
$$\Rightarrow \quad 0 = R_{1212} - A_{22}A_{11} + A_{12}^2$$

Gauss' Golden Theorem!

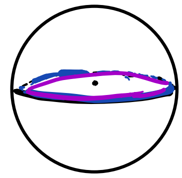
ie $\quad R_{1212} = A_{11}A_{22} - A_{12}^2 = \det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} =: K$ Gauss curvature

Def$^{\underline{n}}$: $M^k \subseteq (\bar{M}^n, \bar{g})$ is **totally geodesic** if $A \equiv 0$ at every $p \in M$

E.g.) $(\mathbb{R}^n, g_{Eucl.})$
$\cup\!|$
$\mathbb{R}^{n-1}$
totally geodesic hypersurface



(S^n, g_{round})
$\cup\!|$ totally geodesic hypersurface



$\leftarrow S^{n-1}$

Prop: $M^k \subseteq (\bar{M}^n, \bar{g})$ totally geodesic $\iff$ every geodesics in M are geodesics in $\bar{M}$.

Pf: For any smooth curve $\gamma : I \to M$,

$$\bar{\nabla}_{\gamma'}\gamma' = (\bar{\nabla}_{\gamma'}\gamma')^T + (\bar{\nabla}_{\gamma'}\gamma')^N$$

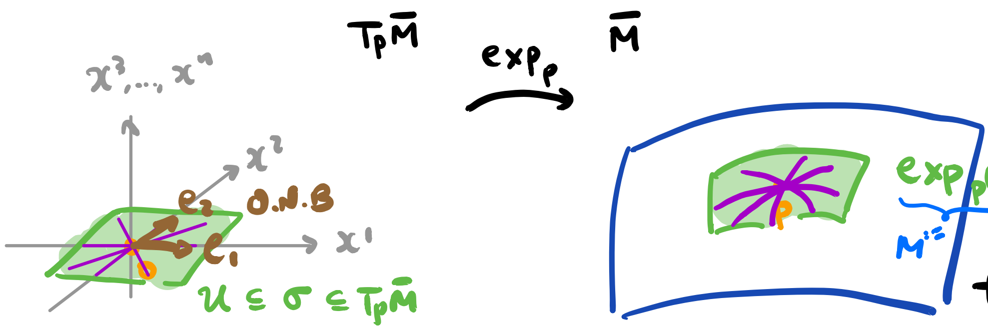
$$= \nabla_{\gamma'}\gamma' + \underbrace{A(\gamma', \gamma')}_{\substack{\equiv 0 \\ \because \text{ totally geodesic}}}$$

$\blacksquare$

Note: Geodesics in $\bar{M}$ lying inside M are always geodesic in M

This gives a geometric interpretation of "Sectional curvature" in terms of "Gauss curvature" for surfaces.

Recall: In geodesic normal coord. centered at $p \in (\bar{M}^n, \bar{g})$.



$\exp_p(\mathcal{U})$ is a 2-dim'l surface which is totally geodesic at p

By Gauss eq$^{\underline{n}}$, at $p \in M^2 \subseteq (\bar{M}^n, \bar{g})$.

$$\underbrace{\bar{R}(e_1, e_2, e_1, e_2)}_{\bar{K}_p(\sigma)} = \underbrace{R(e_1, e_2, e_1, e_2)}_{\substack{\| \text{ Gauss curvature} \\ K_M(p)}} + \underbrace{(\text{quadratic terms of } A)}^{=0 \text{ at } p}$$

Remark: Totally geodesic submanifolds rarely exists in general $(\bar{M}^n, \bar{g})$.

We want to define a "weaker" notion.

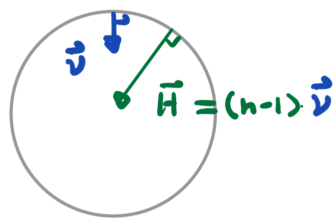
Def$^{\underline{n}}$: $M^k \subseteq (\bar{M}^n, \bar{g})$ $\rightsquigarrow$ The mean curvature vector at $p \in M$

$$\vec{H}(p) := \sum_{i=1}^{k} A_p(e_i, e_i) \qquad \text{where } \{e_1, \dots, e_k\} \text{ O.N.B. for } T_pM$$

And we say M is minimal if $\vec{H} \equiv \vec{0}$ at every $p \in M$

Remarks: 1) totally geodesic $\overset{\Longrightarrow}{\not\Longleftarrow}$ minimal

E.g.) $S^{n-1} \subseteq \mathbb{R}^n$



2) In codim. 1 case, we write

$$\vec{H} = H \vec{\nu}$$

H: (scalar) mean curvature (sign depends on choice of $\vec{\nu}$)

$\vec{\nu}$: unit normal

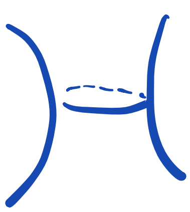
3) When $k = \dim M = 1$, then "minimal" $\Leftrightarrow$ "geodesic"

In fact, minimal k-submanifolds are critical points to the k-dim'l area functional, just like "geodesics" are critical points to the length functional.

E.g.) Minimal surfaces in $\mathbb{R}^3$

plane

totally geodesic



catenoid

helicoid

∞'ly many more

$\circ\circ\circ \; \circ\circ\circ$

Minimal surfaces in $(\mathbb{S}^3, g_{round})$

great sphere
totally geodesic

Clifford torus
(HW)

Lawson surfaces
($\sim$1970's)

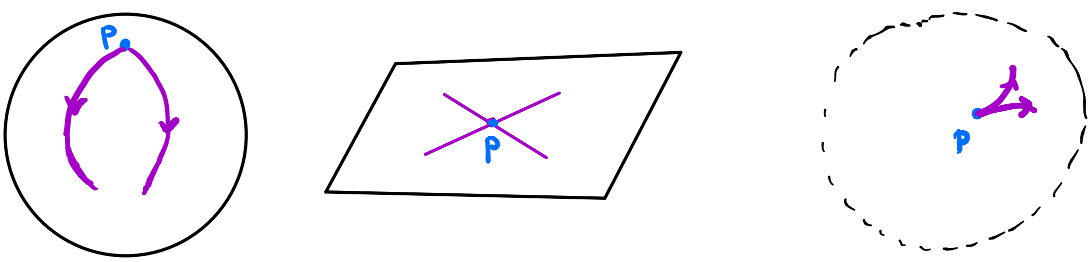
∞'ly many
$\circ\circ\circ\circ\circ\circ$

Geodesics & Jacobi Fields

Q: Given (M^n, g), how does "curvatures" affect "geometry"?

Recall: The effect of "Gauss curvature" on geodesics in surfaces

$K > 0$ $(\mathbb{S}^2)$ $K = 0$ $(\mathbb{R}^2)$ $K < 0$ $(\mathbb{H}^2)$



geodesic "converges"

geodesic "diverges"

Q: What about in higher dim'l ?

"**A**": "curvatures" affect the "stability" of geodesics

or more general, of minimal submanifolds.

$\rightsquigarrow$ 1^{st} & 2^{nd} variation for length/energy functional on curves!

1^{st} & 2^{nd} Variation Formula (for length / energy)

Def$^{\underline{n}}$: Given a (piecewise) smooth curve $\gamma : [a,b] \to (M,g)$

define **Length** $L(\gamma) := \int_a^b \sqrt{g(\gamma'(t),\gamma'(t))}\,dt \leftarrow$ (indep. of reparametrization)

Energy $E(\gamma) := \frac{1}{2}\int_a^b g(\gamma'(t),\gamma'(t))\,dt \leftarrow$ (depending on parametrize)

Remark: By Hölder inequality

$$L(\gamma) \le \sqrt{2}\sqrt{b-a}\; E(\gamma)^{\frac{1}{2}}$$

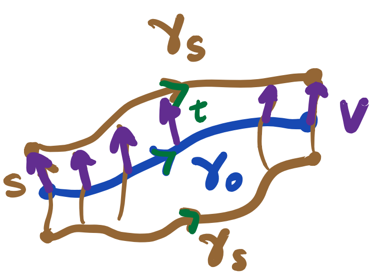
and "=" holds $\Longleftrightarrow$ $\|\gamma'(t)\| := \sqrt{g(\gamma'(t),\gamma'(t))} \equiv$ const

$\Rightarrow$ L & E are the same (up to a multiplicative constant)

for curves parametrized proportional to arc length.

Setup: Consider a 1-parameter family of smooth curves in (M,g)

$$\gamma(t,s) := \gamma_s(t) : \overbrace{[a,b]}^{t} \times \overbrace{(-\varepsilon,\varepsilon)}^{s} \longrightarrow M \qquad \text{smooth}$$



Look at the function of s

$$L(s) := L(\gamma_s)$$

$$E(s) := E(\gamma_s)$$

Goal: Compute $L'(0)$, $L''(0)$ and $E'(0)$, $E''(0)$

Notation: write the variation vector field as

$$V(t) := \frac{\partial\gamma}{\partial s}(t,0) \qquad \text{a vector field along } \gamma_0$$

We start with the 1^{st} variation.

1^{st} variation formula:

$$E'(s) = -\int_a^b \left< \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right> dt + \left< \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right> \Big|_{t=a}^{t=b}$$

and $$L'(s) = \int_a^b \frac{1}{\left\| \frac{\partial \gamma}{\partial t} \right\|} \left(\frac{\partial}{\partial t} \left< \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right> - \left< \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right> \right) dt$$

Proof: $E(s) := E(\gamma_s) := \frac{1}{2} \int_a^b \left< \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right> dt$

$$\Rightarrow E'(s) = \frac{d}{ds} \left(\frac{1}{2} \int_a^b \left< \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right> dt \right)$$

$$= \frac{1}{2} \int_a^b \frac{\partial}{\partial s} \left< \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right> dt$$

metric compatible $\rightarrow$ $$= \int_a^b \left< \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right> dt$$

torsion-free
Recall: $\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0$ $\rightarrow$ $$= \int_a^b \left< \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right> dt$$

metric compatible $\rightarrow$ $$= \int_a^b \left(\frac{\partial}{\partial t} \left< \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right> - \left< \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right> \right) dt$$

$$= -\int_a^b \left< \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right> dt + \left< \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right> \Big|_{t=a}^{t=b}$$

Similarly, $L(s) := \int_a^b \sqrt{\left< \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right>} \, dt$

$$\Rightarrow L'(s) = \int_a^b \frac{1}{\left\| \frac{\partial \gamma}{\partial t} \right\|} \left< \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right> dt$$

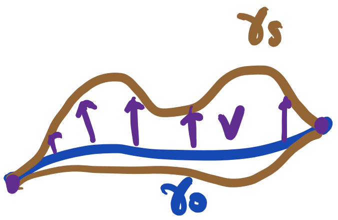
$$= \int_a^b \frac{1}{\left\| \frac{\partial \gamma}{\partial t} \right\|} \left(\frac{\partial}{\partial t} \left< \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right> - \left< \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right> \right) dt$$

$\blacksquare$

If the end points are fixed in the variation:

ie $\gamma_s(a) = \gamma_0(a)$ & $\gamma_s(b) = \gamma_0(b)$ $\forall s \in (-\varepsilon, \varepsilon)$

then $V(a) = 0 = V(b)$ and



$$E'(0) = -\int_a^b \left\langle V(t), \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma_0}{\partial t} \right\rangle dt$$

$$L'(0) = -\frac{1}{\left\| \frac{\partial \gamma_0}{\partial t} \right\|} \int_a^b \left\langle V(t), \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma_0}{\partial t} \right\rangle dt$$

provided that $\left\| \frac{\partial \gamma_0}{\partial t} \right\| \equiv$ const.

Cor: crit. pts of E $\iff$ $\nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma_0}{\partial t} \equiv 0$, ie. γ_0 is geodesic

crit. pts of L $\iff$ $\nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma_0}{\partial t} \equiv 0$, ie. γ_0 is geodesic

provided that $\left\| \frac{\partial \gamma_0}{\partial t} \right\| \equiv$ const.

Next, we assume γ_0 is a geodesic, ie $\nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma_0}{\partial t} \equiv 0$

and compute the 2nd variation, with end pts fixed.

2nd variation formula:

$$E''(0) = \int_a^b \left(\left\langle \nabla_{\frac{\partial}{\partial t}} V, \nabla_{\frac{\partial}{\partial t}} V \right\rangle - \left\langle R\left(\frac{\partial \gamma_0}{\partial t}, V\right) \frac{\partial \gamma_0}{\partial t}, V \right\rangle \right) dt$$

$$L''(0) = \frac{1}{\left\| \frac{\partial \gamma_0}{\partial t} \right\|} \int_a^b \left(\left\langle \nabla_{\frac{\partial}{\partial t}} V^N, \nabla_{\frac{\partial}{\partial t}} V^N \right\rangle - \left\langle R\left(\frac{\partial \gamma_0}{\partial t}, V^N\right) \frac{\partial \gamma_0}{\partial t}, V^N \right\rangle \right) dt$$